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## Finite plane deformations of a three-phase circular inhomogeneity-matrix system

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## ABSTRACT

We consider finite plane deformations of a three-phase circular inhomogeneity-matrix system in which the inhomogeneity, the interphase layer and the matrix belong to the same class of compressible hyperelastic materials of harmonic-type but with each phase possessing its own distinct material properties. We obtain the complete solution when the system is subjected to general classes of remote (Piola) stress, specifically, remote stress distributions characterized by stress functions described by general polynomials of order  $n \geq 1$  in the corresponding complex variable  $z$  used to describe the matrix. As a particular case of the aforementioned analysis, we establish an Eshelby-type result namely that, for this class of harmonic materials, a three-phase circular inhomogeneity under uniform remote stress and eigenstrain, admits an internal uniform stress field when subjected to plane deformations.

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## 1. Introduction

Composite materials involving elastic inhomogeneities have consistently been the subject of intense research mainly because of their practical applications in materials science and engineering. Most theoretical analyses of the corresponding problems, however, have been confined to linear elasticity in which the composite structures are subjected only to infinitesimal deformations (see, for example, [1] for an extensive bibliography). This can be attributed primarily to the fact that, in the more practical case of *finite* deformations, the nonlinear nature of the ensuing governing equations have inhibited similar theoretical investigations. Works by Ogden and Isherwood [2] and Varley and Cumberbatch [3], however, have provided complex-variable formulations of a class of problems involving the (finite) plane-strain deformations of a set of compressible hyperelastic materials of harmonic-type, originally proposed by John [4]. These materials have been found to be of significant practical and theoretical interest (see, for example, [5–12]). More recently, Ru [13] has further developed the complex variable formulation presented in [3] and obtained a version particularly suitable for the study of problems involving elastic inhomogeneities. Introductory problems involving elastic inhomogeneities for this class of nonlinear (harmonic) materials have been solved in [2] and [14].

In [15], the authors continued the work begun in [14] by considering plane finite deformations of a composite material in which a circular elastic inhomogeneity is embedded in the same class of harmonic materials under the assumption of *nonuniform* remote loading in the surrounding matrix. These results are important in that they essentially lead to solutions of problems in which the inhomogeneity-matrix system is subjected to a wide class of remote loading conditions and so accommodate a correspondingly large set of physically relevant problems. The results in [15], however, are limited to the case when the inhomogeneity is assumed to be perfectly bonded to the matrix. As noted in [1], this assumption is

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unrealistic for practical purposes basically because it ignores the presence of a real interphase layer which may include, for example, the effects of voids, cracks and regions of imperfect adhesion, all of which may significantly affect the ensuing stress distributions in the composite structure.

In this paper, we generalize the model used in [15] by adding a third intermediate interphase layer between the inhomogeneity and the surrounding matrix. This layer is assumed to have its own material properties and, as such, can be used as a first step in the modelling of effects not accommodated by the two-phase, perfect interface model [14,15]. We obtain the complete solution when the three-phase system is subjected to classes of nonuniform remote stresses characterized by stress functions described by general polynomials of degree  $n$ . These solutions are useful in that they aid in the understanding of the effect of an intermediate layer between the inhomogeneity and its surrounding matrix and its ensuing effect on the stress distributions inside the inhomogeneity (which represents a cross-section of a fibre embedded in a composite material) [16]. As a particular case of the aforementioned analysis, we obtain an Eshelby-type result that a three-phase circular inhomogeneity under uniform remote stress and eigenstrain, admits an internal uniform stress field when subjected to plane (finite) deformations. As a general reference to classical applications of complex variable methods in nonlinear elasticity, the reader is referred to [17,18].

## 2. Notation and prerequisites

Let  $z = x_1 + ix_2$  be the initial coordinates of a material particle in the undeformed configuration and  $w(z) = y_1(z) + iy_2(z)$ , the corresponding spatial coordinates in the deformed configuration. The (Cartesian) components of the deformation gradient tensor are given by:

$$F_{ij} = \frac{\partial y_i}{\partial x_j} = y_{i,j}.$$

We consider the particular class of harmonic materials, discussed in detail in [10,11] whose strain energy density  $W$  is characterized by

$$W = 2\mu[F(I) - J], \quad F'(I) = \frac{1}{4\alpha}[I + \sqrt{I^2 - 16\alpha\beta}], \quad (1)$$

where [12],

$$I = \sqrt{F_{ij}F_{ij} + 2J}, \quad J = \det[F_{ij}].$$

$\mu$  is the shear modulus and  $\alpha, \beta$  are two material constants defined explicitly in [13] (note that the function  $F(I)$  is a material function of  $I$  and is not to be confused with the components of the deformation gradient tensor  $\mathbf{F}$ ).

Consider a domain in  $\mathbb{R}^2$ , infinite in extent, containing a single internal circular inhomogeneity. The inhomogeneity is assumed to be surrounded by an interphase layer of finite thickness. The (unbounded) exterior of the interphase layer is referred to as the matrix. Let  $S_0, S_1$  and  $S_2$  denote the inhomogeneity, the interphase layer and the matrix, respectively, all of which are perfectly bonded through two concentric circles  $R_i$  and  $R_o$ , as shown in Fig. 2.1.

In what follows, the subscripts 0, 1 and 2 will refer to the regions  $S_0, S_1$  and  $S_2$ , respectively. The elastic materials occupying the inhomogeneity, the interphase layer and the matrix belong to the class of harmonic materials characterized by Eq. (1) above with associated elastic constants  $\mu_\gamma, \alpha_\gamma, \beta_\gamma$ ,  $\gamma = 0, 1, 2$ . From [13], the (plane) deformation function  $w$  in the inhomogeneity, the interphase layer and in the matrix can be written, respectively, in terms of two analytic functions  $\phi$  and  $\psi$  as:

$$iw_\gamma(z) = \alpha_\gamma \phi_\gamma(z) + i\overline{\psi_\gamma(z)} + \frac{\beta_\gamma z}{\phi'_\gamma(z)}, \quad \gamma = 0, 1, 2 \text{ (no sum over repeated indices)}. \quad (2)$$

Similarly, the complex Piola stress function [18]  $\chi$  can be written in terms of  $\phi$  and  $\psi$  in the inhomogeneity, the interphase layer and the matrix as:

$$\chi_\gamma(z) = 2\mu_\gamma i \left[ (\alpha_\gamma - 1)\phi_\gamma(z) + i\overline{\psi_\gamma(z)} + \frac{\beta_\gamma z}{\phi'_\gamma(z)} \right], \quad \gamma = 0, 1, 2 \text{ (no sum over repeated indices)}. \quad (3)$$

Continuity conditions for the displacement and Piola stress across each (perfectly) bonded circular boundary  $R_i$  and  $R_o$  can be written as follows.

For the inner boundary ( $|z| = R_i$ ):

$$\alpha_0 \phi_0(z) + i\overline{\psi_0(z)} + \frac{\beta_0 z}{\phi'_0(z)} + iu^*z = \alpha_1 \phi_1(z) + i\overline{\psi_1(z)} + \frac{\beta_1 z}{\phi'_1(z)}, \quad (4)$$

$$\mu_0 \left[ (\alpha_0 - 1)\phi_0(z) + i\overline{\psi_0(z)} + \frac{\beta_0 z}{\phi'_0(z)} \right] = \mu_1 \left[ (\alpha_1 - 1)\phi_1(z) + i\overline{\psi_1(z)} + \frac{\beta_1 z}{\phi'_1(z)} \right]. \quad (5)$$

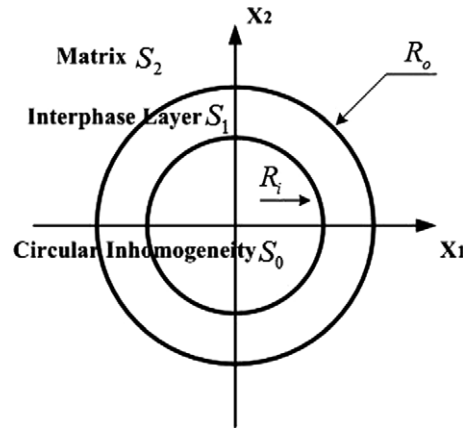


Fig. 2.1. Three-phase circular inhomogeneity.

For the outer boundary ( $|z| = R_o$ ):

$$\alpha_1 \phi_1(z) + i \overline{\psi_1(z)} + \frac{\beta_1 z}{\phi'_1(z)} = \alpha_2 \phi_2(z) + i \overline{\psi_2(z)} + \frac{\beta_2 z}{\phi'_2(z)}, \quad (6)$$

$$\mu_1 \left[ (\alpha_1 - 1) \phi_1(z) + i \overline{\psi_1(z)} + \frac{\beta_1 z}{\phi'_1(z)} \right] = \mu_2 \left[ (\alpha_2 - 1) \phi_2(z) + i \overline{\psi_2(z)} + \frac{\beta_2 z}{\phi'_2(z)} \right], \quad (7)$$

where  $u^*$  is a stress-free displacement caused by uniform eigenstrains imposed on the circular inhomogeneity  $S_0$ .

**Remark 1.** From [13], we note that, for the class of harmonic materials considered here, a state of zero displacement is characterized by

$$1 = |\phi'_\gamma(z)| = F'(I),$$

$$i w_\gamma(z) = \alpha_\gamma \phi_\gamma(z) + i \overline{\psi_\gamma(z)} + \frac{\beta_\gamma z}{\phi'_\gamma(z)} = iz,$$

which in turn leads to

$$\phi_\gamma(z) = iz, \quad \psi_\gamma(z) = 0, \quad \gamma = 0, 1, 2.$$

In view of Remark 1, we consider problems corresponding to cases in which the three-phase circular inhomogeneity-matrix system is subjected to a general class of prescribed remote loadings described by analytic functions  $\phi_2$  and  $\psi_2$  of the (polynomial) form:

$$\phi_2(z) = A_1 z + \sum_{n=a}^b A_n z^n, \quad \psi_2(z) = B_1 z + \sum_{n=a}^b B_n z^n, \quad |z| \rightarrow \infty, \quad a \geq 2, \quad b \geq a, \quad (8)$$

where  $A_1, A_n, B_1$  and  $B_n$  are known complex constants.

This class can accommodate a wide range of applied remote Piola stress distributions.

### 3. Complete solution of the three-phase circular inhomogeneity

The form of remote loading (8) suggests that we seek solutions corresponding to the following nonuniform Piola stress distributions inside the inhomogeneity:

$$\phi_0(z) = C_1 z + \sum_{n=a}^b C_n z^n, \quad \psi_0(z) = D_1 z + \sum_{n=a}^b D_n z^n, \quad |z| < R_i, \quad a \geq 2, \quad b \geq a, \quad (9)$$

where  $C_1, C_n, D_1, D_n$  are complex constants to be determined and  $R_i$  is the radius of the (circular) inhomogeneity.

First, we note that the interface conditions (4)–(7) can be written in the form

$$\begin{aligned} \phi_1(z) &= \phi_0(z) K_1 + i S_1 \overline{\psi_0(z)} + \frac{S_1 \beta_0 z}{\phi'_0(z)} + i u^* z, \\ \psi_1(z) &= i \left[ \alpha_0 \overline{\phi_0(z)} + \frac{\beta_0 \bar{z}}{\phi'_0(z)} - \alpha_1 \phi_1(z) - \frac{\beta_1 \bar{z}}{\phi'_1(z)} \right] + \psi_0(z) + u^* \bar{z}, \quad |z| = R_i, \end{aligned} \quad (10)$$

$$\begin{aligned}\phi_2(z) &= \phi_1(z)K_2 + iS_2\overline{\psi_1(z)} + \frac{S_2\beta_1 z}{\phi_1'(z)}, \\ \psi_2(z) &= i\left[\alpha_1\overline{\phi_1(z)} + \frac{\beta_1\bar{z}}{\phi_1'(z)} - \alpha_2\overline{\phi_2(z)} - \frac{\beta_2\bar{z}}{\phi_2'(z)}\right] + \psi_1(z), \quad |z| = R_0,\end{aligned}\quad (11)$$

where the constants  $K_1$ ,  $K_2$ ,  $S_1$ ,  $S_2$  are defined by  $K_1 = \alpha_0(1 - \mu_0/\mu_1) + \mu_0/\mu_1$ ,  $K_2 = \alpha_1(1 - \mu_1/\mu_2) + \mu_1/\mu_2$ ,  $S_1 = 1 - \mu_0/\mu_1$ , and  $S_2 = 1 - \mu_1/\mu_2$ .

**Remark 2.** Since, it is required that [13]

$$|\phi_\gamma'(z)| = F'(I) \neq 0, \quad \gamma = 0, 1, 2,$$

it is clear that we must have

$$A_1, C_1 \neq 0, \\ C_1 + \sum_{n=a}^b nC_n z^{n-1} \neq 0 \iff \overline{C_1} + \sum_{n=a}^b n\overline{C_n} \overline{z^{n-1}} \neq 0, \quad \forall z: |z| \leq R_i.$$

From (9) and (10), we now have that

$$\phi_1(z) = K_1 \left[ C_1 z + \sum_{n=a}^b C_n z^n \right] + iS_1 \left[ \overline{D_1} \bar{z} + \sum_{n=a}^b \overline{D_n} \bar{z}^n \right] + \frac{S_1 \beta_0 z}{\overline{C_1} + \sum_{n=a}^b n\overline{C_n} \overline{z^{n-1}}} + iu^* z, \quad |z| = R_i. \quad (12)$$

Since  $z\bar{z} = |z|^2 = R_i^2$ , at the boundary of the circular inhomogeneity, we obtain, for  $|z| = R_i$ :

$$\begin{aligned}\phi_1(z) &= K_1 C_1 z + \sum_{n=a}^b (K_1 C_n z^n) + \frac{iS_1 \overline{D_1} R_i^2}{z} \\ &\quad + \sum_{n=a}^b \left( \frac{iS_1 \overline{D_n} R_i^{2n}}{z^n} \right) + \frac{S_1 \beta_0 z}{\overline{C_1} + \sum_{n=a}^b \left( \frac{n\overline{C_n} R_i^{2(n-1)}}{z^{n-1}} \right)} + iu^* z.\end{aligned}\quad (13)$$

We adopt the form  $\phi_1$  from (13) for the extended region  $R_0 > |z| > R_i$ , so that

$$\phi_1(z) = X_1 z + \sum_{n=a}^b X_n^1 z^n + \frac{Y_1}{z} + \sum_{n=a}^b \frac{Y_n^1}{z^n} + R(z), \quad R_0 > |z| > R_i, \quad (14)$$

where  $X_1 = K_1 C_1 + iu^*$ ,  $X_n^1 = K_1 C_n$ ,  $Y_1 = iS_1 \overline{D_1} R_i^2$ ,  $Y_n^1 = iS_1 \overline{D_n} R_i^{2n}$ ,

$$R(z) = \frac{S_1 \beta_0 z}{\overline{C_1} + \sum_{n=a}^b \left( \frac{n\overline{C_n} R_i^{2(n-1)}}{z^{n-1}} \right)}.$$

**Remark 3.** As a direct consequence of Remark 2, we have that

$$X_1 + \sum_{n=a}^b (nX_n^1 z^{n-1}) - \frac{Y_1}{z^2} - \sum_{n=a}^b \left( \frac{nY_n^1}{z^{n+1}} \right) + R'(z) \neq 0, \quad \forall z: R_0 \geq |z| \geq R_i.$$

From the interface conditions (4) and (5) (or (10)), (9) and (14), it follows that, for  $|z| = R_i$ ,

$$\begin{aligned}\psi_1(z) &= i\left(\alpha_0\overline{\phi_0(z)} + \frac{\beta_0\bar{z}}{\phi_0'(z)} - \alpha_1\overline{\phi_1(z)} - \frac{\beta_1\bar{z}}{\phi_1'(z)}\right) + \psi_0(z) + u^* \bar{z} \\ &= -i\alpha_1 \left[ \frac{\overline{X_1} R_i^2}{z} + \sum_{n=a}^b \left( \frac{\overline{X_n^1} R_i^{2n}}{z^n} \right) + \frac{\overline{Y_1}}{R_i^2} z + \sum_{n=a}^b \left( \frac{\overline{Y_n^1}}{R_i^{2n}} z^n \right) + \frac{S_1 \beta_0 R_i^2}{z[\sum_{n=a}^b (nC_n z^{n-1}) + C_1]} \right] \\ &\quad - \frac{i\beta_1 R_i^2}{z[X_1 + \sum_{n=a}^b (nX_n^1 z^{n-1}) - \frac{Y_1}{z^2} - \sum_{n=a}^b \left( \frac{nY_n^1}{z^{n+1}} \right) + R'(z)]} + i\alpha_0 \left[ \frac{\overline{C_1} R_i^2}{z} + \sum_{n=a}^b \left( \frac{\overline{C_n} R_i^{2n}}{z^n} \right) \right] \\ &\quad + \frac{i\beta_0 R_i^2}{z(C_1 + \sum_{n=a}^b nC_n z^{n-1})} + D_1 z + \sum_{n=a}^b D_n z^n + \frac{u^* R_i^2}{z}.\end{aligned}\quad (15)$$

If we adopt the form  $\psi_1$  from (15) for the region  $R_o > |z| > R_i$ , we have

$$\psi_1(z) = P_1 z + \sum_{n=a}^b P_n^1 z^n + \frac{Q_1}{z} + \sum_{n=a}^b \frac{Q_n^1}{z^n} + H(z), \quad R_o > |z| > R_i, \quad (16)$$

where

$$\begin{aligned} H(z) &= \frac{i\beta_0 R_i^2 (1 - \alpha_1 S_1)}{z(C_1 + \sum_{n=a}^b n C_n z^{n-1})} - \frac{i\beta_1 R_i^2}{z[X_1 + \sum_{n=a}^b (n X_n^1 z^{n-1}) - \frac{Y_1}{z^2} - \sum_{n=a}^b (\frac{n Y_n^1}{z^{n+1}}) + R'(z)]}, \\ P_1 &= \frac{-i\alpha_1 \bar{Y}_1}{R_i^2} + D_1, \quad Q_1 = i\alpha_0 \bar{C}_1 R_i^2 - i\alpha_1 \bar{X}_1 R_i^2 + u^* R_i^2, \\ P_n^1 &= \frac{-i\alpha_1 \bar{Y}_n^1}{R_i^{2n}} + D_n, \quad Q_n^1 = i\alpha_0 \bar{C}_n R_i^{2n} - i\alpha_1 \bar{X}_n^1 R_i^{2n}. \end{aligned}$$

Therefore, for the external boundary  $R_o$ , we have from (11), (14) and (16) that

$$\begin{aligned} \phi_2(z) &= \phi_1(z) K_2 + i S_2 \bar{\psi}_1(z) + \frac{S_2 \beta_1 z}{\phi_1'(z)} = K_2 \left[ X_1 z + \sum_{n=a}^b X_n^1 z^n + \frac{Y_1}{z} + \sum_{n=a}^b \frac{Y_n^1}{z^n} + R(z) \right] \\ &\quad + i S_2 \left[ \bar{P}_1 \bar{z} + \sum_{n=a}^b \bar{P}_n^1 \bar{z}^n + \frac{\bar{Q}_1}{\bar{z}} + \sum_{n=a}^b \frac{\bar{Q}_n^1}{\bar{z}^n} + \bar{H}(z) \right] \\ &\quad + \frac{S_2 \beta_1 z}{\bar{X}_1 + \sum_{n=a}^b (n \bar{X}_n^1 \bar{z}^{n-1}) - \frac{\bar{Y}_1}{\bar{z}^2} - \sum_{n=a}^b (\frac{n \bar{Y}_n^1}{\bar{z}^{n+1}}) + \bar{R}'(z)}. \end{aligned}$$

Since,  $z\bar{z} = R_o^2$ , at the external circular boundary, and adopting  $\phi_2(z)$  for the region  $|z| > R_o$ ,

$$\phi_2(z) = X_2 z + \sum_{n=a}^b X_n^2 z^n + \frac{Y_2}{z} + \sum_{n=a}^b \frac{Y_n^2}{z^n} + K_2 R(z) + I(z), \quad |z| > R_o, \quad (17)$$

where

$$\begin{aligned} X_2 &= K_2 X_1 + \frac{i S_2 \bar{Q}_1}{R_o^2}, \quad X_n^2 = K_2 X_n^1 + \frac{i S_2 \bar{Q}_n^1}{R_o^{2n}}, \\ Y_2 &= K_2 Y_1 + i S_2 \bar{P}_1 R_o^2, \quad Y_n^2 = K_2 Y_n^1 + i S_2 \bar{P}_n^1 R_o^{2n}, \\ I(z) &= \frac{(1 - \alpha_1 S_1) S_2 \beta_0 R_i^2 z}{R_o^2 (\bar{C}_1 + \sum_{n=a}^b \frac{n \bar{C}_n R_o^{2(n-1)}}{z^{n-1}})} + \frac{z \beta_1 S_2 (R_o^2 - R_i^2)}{R_o^2 (\bar{X}_1 + \sum_{n=a}^b \frac{n \bar{X}_n^1 R_o^{2(n-1)}}{z^{n-1}} - \frac{\bar{Y}_1 z^2}{R_o^4} - \sum_{n=a}^b \frac{n \bar{Y}_n^1 z^{n+1}}{R_o^{2(n+1)}} + J(z))}, \\ J(z) &= \frac{S_1 \beta_0}{C_1 + \sum_{n=a}^b \frac{n C_n R_i^{2(n-1)}}{R_o^{2(n-1)}}} + \frac{S_1 \beta_0 (\sum_{n=a}^b \frac{n(n-1) C_n R_i^{2(n-1)} z^{n-1}}{R_o^{2(n-1)}})}{(C_1 + \sum_{n=a}^b \frac{n C_n R_i^{2(n-1)}}{R_o^{2(n-1)}})^2}. \end{aligned}$$

Consequently, from (17), we have that

$$\phi_2(z) = X_2 z + \sum_{n=a}^b X_n^2 z^n + \left[ \frac{K_2 S_1 \beta_0}{\bar{C}_1} + \frac{(1 - \alpha_1 S_1) S_2 \beta_0 R_i^2}{R_o^2 \bar{C}_1} \right] z, \quad |z| \rightarrow \infty.$$

Clearly,  $\phi_2(z)$  must also satisfy the remote loading condition (8), so that

$$A_1 z + \sum_{n=a}^b A_n z^n = \left[ X_2 + \frac{K_2 S_1 \beta_0}{\bar{C}_1} + \frac{(1 - \alpha_1 S_1) S_2 \beta_0 R_i^2}{R_o^2 \bar{C}_1} \right] z + \sum_{n=a}^b X_n^2 z^n. \quad (18)$$

Therefore, from (14), (16)–(18), the unknown complex constants  $C_1$  and  $C_n$  are determined by the equations:

$$\begin{aligned} A_1 &= \left( K_2 - \frac{\alpha_1 S_2 R_i^2}{R_o^2} \right) \left[ K_1 C_1 + \frac{S_1 \beta_0}{\bar{C}_1} + i u^* \right] + \frac{S_2 R_i^2}{R_o^2} \left[ \alpha_0 C_1 + \frac{\beta_0}{\bar{C}_1} + i u^* \right], \\ \sum_{n=a}^b A_n z^n &= \sum_{n=a}^b \left[ \left( K_2 - \alpha_1 S_2 \frac{R_i^{2n}}{R_o^{2n}} \right) K_1 C_n + S_2 \alpha_0 C_n \frac{R_i^{2n}}{R_o^{2n}} \right] z^n. \end{aligned} \quad (19)$$

**Remark 4.** As a direct consequence of Remark 2, we have that

$$X_2 + \sum_{n=a}^b (nX_n^2 z^{n-1}) - \frac{Y_2}{z^2} - \sum_{n=a}^b \left( \frac{nY_n^2}{z^{n+1}} \right) + K_2 R'(z) + I'(z) \neq 0, \quad \forall z: |z| \geq R_0.$$

This implies, together with Remark 2 and (19), that there are restrictions on the constants  $A_1$ ,  $A_n$ ,  $B_1$ , and  $B_n$  characterizing the remote stresses.

It remains to determine the analytic function  $\psi_2(z)$ .

From the interface conditions (6) and (7) (or (11)), (14), (16), and (17) it follows that, for  $|z| = R_0$ ,

$$\psi_2(z) = i \left[ \alpha_1 \overline{\phi_1(z)} + \frac{\beta_1 \bar{z}}{\phi_1'(z)} - \alpha_2 \overline{\phi_2(z)} - \frac{\beta_2 \bar{z}}{\phi_2'(z)} \right] + \psi_1(z),$$

and evaluating  $\overline{R(z)}$  and  $\overline{I(z)}$  on  $|z| = R_0$ , we have that

$$\begin{aligned} \psi_2(z) = i & \left[ \alpha_1 \left( \frac{\overline{X_1} R_0^2}{z} + \sum_{n=a}^b \left( \frac{\overline{X_n^1} R_0^{2n}}{z^n} \right) + \frac{\overline{Y_1} z}{R_0^2} + \sum_{n=a}^b \left( \frac{\overline{Y_n^1} z^n}{R_0^{2n}} \right) + \overline{R(z)} \right) \right. \\ & + \frac{\beta_1 R_0^2}{z(X_1 + \sum_{n=a}^b (nX_n^1 z^{n-1}) - \frac{Y_1}{z^2} - \sum_{n=a}^b (\frac{nY_n^1}{z^{n+1}}) + R'(z))} \\ & - \alpha_2 \left( \frac{\overline{X_2} R_0^2}{z} + \sum_{n=a}^b \frac{\overline{X_n^2} R_0^{2n}}{z^n} + \frac{\overline{Y_2} z}{R_0^2} + \sum_{n=a}^b \frac{\overline{Y_n^2} z^n}{R_0^{2n}} + K_2 \overline{R(z)} + \overline{I(z)} \right) \\ & \left. - \frac{\beta_2 R_0^2}{z(X_2 + \sum_{n=a}^b (nX_n^2 z^{n-1}) - \frac{Y_2}{z^2} - \sum_{n=a}^b (\frac{nY_n^2}{z^{n+1}}) + K_2 R'(z) + I'(z))} \right] \\ & + P_1 z + \sum_{n=a}^b P_n^1 z^n + \frac{Q_1}{z} + \sum_{n=a}^b \frac{Q_n^1}{z^n} + H(z), \end{aligned} \quad (20)$$

where

$$\begin{aligned} \overline{R(z)} &= \frac{S_1 \beta_0 R_0^2}{z(C_1 + \sum_{n=a}^b (\frac{nC_n R_i^{2(n-1)}}{R_0^{2(n-1)}}) z^{n-1})}, \\ \overline{I(z)} &= \frac{(1 - \alpha_1 S_1) S_2 \beta_0 R_i^2}{z(C_1 + \sum_{n=a}^b nC_n z^{n-1})} + \frac{S_2 \beta_1 (R_0^2 - R_i^2)}{z(X_1 + \sum_{n=a}^b nX_n^1 z^{n-1} - \frac{Y_1}{z^2} - \sum_{n=a}^b \frac{nY_n^1}{z^{n+1}} + \overline{J(z)}),} \\ \overline{J(z)} &= \frac{S_1 \beta_0}{\overline{C_1} + \sum_{n=a}^b \frac{n\overline{C_n} R_i^{2(n-1)}}{z^{n-1}}} + \frac{S_1 \beta_0 (\sum_{n=a}^b \frac{(n-1)n\overline{C_n} R_i^{2(n-1)}}{z^{n-1}})}{(\overline{C_1} + \sum_{n=a}^b \frac{n\overline{C_n} R_i^{2(n-1)}}{z^{n-1}})^2}. \end{aligned}$$

From (8),  $\psi_1$  must also satisfy the asymptotic condition

$$\psi_2(z) = B_1 z + \sum_{n=a}^b B_n z^n, \quad |z| \rightarrow \infty. \quad (21)$$

Hence, if we adopt  $\psi_1$  from (20) for  $|z| > R_0$ , we require that

$$\left( i\alpha_1 \frac{\overline{Y_1}}{R_0^2} - i\alpha_2 \frac{\overline{Y_2}}{R_0^2} + P_1 \right) z + \sum_{n=a}^b \left[ i\alpha_1 \frac{\overline{Y_n^1}}{R_0^{2n}} - i\alpha_2 \frac{\overline{Y_n^2}}{R_0^{2n}} + P_n^1 \right] z^n = B_1 z + \sum_{n=a}^b B_n z^n. \quad (22)$$

Thus, from (14), (16), and (17), we require that

$$\begin{aligned} B_1 &= \frac{S_1 D_1 R_i^2}{R_0^2} (\alpha_1 - \alpha_2 K_2) + D_1 (1 - \alpha_2 S_2) (1 - \alpha_1 S_1), \\ \sum_{n=a}^b B_n z^n &= \sum_{n=a}^b \left[ \left( \frac{R_i}{R_0} \right)^{2n} S_1 D_n (\alpha_1 - \alpha_2 K_2) + D_n (1 - \alpha_2 S_2) (1 - \alpha_1 S_1) z^n \right]. \end{aligned} \quad (23)$$

### 3.1. Summary

The complete solution of the inhomogeneity-matrix system subjected to classes of remote loading described by (8) is given by

$$\phi_0(z) = C_1 z + \sum_{n=a}^b C_n z^n, \quad \psi_0(z) = D_1 z + \sum_{n=a}^b D_n z^n, \quad |z| < R_i, \quad a \geq 2, \quad b \geq a, \quad (24)$$

$$\phi_1(z) = X_1 z + \sum_{n=a}^b X_n^1 z^n + \frac{Y_1}{z} + \sum_{n=a}^b \frac{Y_n^1}{z^n} + R(z), \quad R_0 > |z| > R_i, \quad (25)$$

$$\psi_1(z) = P_1 z + \sum_{n=a}^b P_n^1 z^n + \frac{Q_1}{z} + \sum_{n=a}^b \frac{Q_n^1}{z^n} + H(z), \quad R_0 > |z| > R_i, \quad (26)$$

$$\phi_2(z) = X_2 z + \sum_{n=a}^b X_n^2 z^n + \frac{Y_2}{z} + \sum_{n=a}^b \frac{Y_n^2}{z^n} + K_2 R(z) + I(z), \quad |z| > R_0, \quad (27)$$

and

$$\begin{aligned} \psi_2(z) = & i \left[ \alpha_1 \left( \frac{\overline{X_1} R_0^2}{z} + \sum_{n=a}^b \left( \frac{\overline{X_n^1} R_0^{2n}}{z^n} \right) + \frac{\overline{Y_1} z}{R_0^2} + \sum_{n=a}^b \left( \frac{\overline{Y_n^1} z^n}{R_0^{2n}} \right) + \overline{R(z)} \right) \right. \\ & + \frac{\beta_1 R_0^2}{z(X_1 + \sum_{n=a}^b (n X_n^1 z^{n-1}) - \frac{Y_1}{z^2} - \sum_{n=a}^b (\frac{n Y_n^1}{z^{n+1}}) + R'(z))} \\ & - \alpha_2 \left( \frac{\overline{X_2} R_0^2}{z} + \sum_{n=a}^b \frac{\overline{X_n^2} R_0^{2n}}{z^n} + \frac{\overline{Y_2} z}{R_0^2} + \sum_{n=a}^b \frac{\overline{Y_n^2} z^n}{R_0^{2n}} + K_2 \overline{R(z)} + \overline{I(z)} \right) \\ & \left. - \frac{\beta_2 R_0^2}{z(X_2 + \sum_{n=a}^b (n X_n^2 z^{n-1}) - \frac{Y_2}{z^2} - \sum_{n=a}^b (\frac{n Y_n^2}{z^{n+1}}) + K_2 R'(z) + I'(z))} \right] \\ & + P_1 z + \sum_{n=a}^b P_n^1 z^n + \frac{Q_1}{z} + \sum_{n=a}^b \frac{Q_n^1}{z^n} + H(z), \end{aligned} \quad (28)$$

where  $|z| > R_0$ . Here, the unknown complex constants  $C_1, C_n, D_1$  and  $D_n$  are determined by the equations

$$\begin{aligned} A_1 = & \left( K_2 - \frac{\alpha_1 S_2 R_i^2}{R_0^2} \right) \left[ K_1 C_1 + \frac{S_1 \beta_0}{\overline{C_1}} + i u^* \right] + \frac{S_2 R_i^2}{R_0^2} \left[ \alpha_0 C_1 + \frac{\beta_0}{\overline{C_1}} + i u^* \right], \\ \sum_{n=a}^b A_n z^n = & \sum_{n=a}^b \left[ \left( K_2 - \alpha_1 S_2 \frac{R_i^{2n}}{R_0^{2n}} \right) K_1 C_n + S_2 \alpha_0 C_n \frac{R_i^{2n}}{R_0^{2n}} \right] z^n, \end{aligned} \quad (29)$$

$$\begin{aligned} B_1 = & \frac{S_1 D_1 R_i^2}{R_0^2} (\alpha_1 - \alpha_2 K_2) + D_1 (1 - \alpha_2 S_2) (1 - \alpha_1 S_1), \\ \sum_{n=a}^b B_n z^n = & \sum_{n=a}^b \left[ \left( \frac{R_i}{R_0} \right)^{2n} S_1 D_n (\alpha_1 - \alpha_2 K_2) + D_n (1 - \alpha_2 S_2) (1 - \alpha_1 S_1) z^n \right], \end{aligned} \quad (30)$$

where

$$\begin{aligned} K_1 = & \alpha_0 \left( 1 - \frac{\mu_0}{\mu_1} \right) + \frac{\mu_0}{\mu_1}, \quad K_2 = \alpha_1 \left( 1 - \frac{\mu_1}{\mu_2} \right) + \frac{\mu_1}{\mu_2}, \\ S_1 = & 1 - \frac{\mu_0}{\mu_1}, \quad S_2 = 1 - \frac{\mu_1}{\mu_2}. \end{aligned}$$

Also, from Remarks 2–4, we must have that

$$\begin{aligned} A_1, C_1 \neq & 0, \\ C_1 + \sum_{n=a}^b n C_n z^{n-1} \neq & 0 \iff \overline{C_1} + \sum_{n=a}^b n \overline{C_n} \overline{z^{n-1}} \neq 0, \quad \forall z: |z| \leq R_i, \end{aligned}$$

$$X_1 + \sum_{n=a}^b (nX_n^1 z^{n-1}) - \frac{Y_1}{z^2} - \sum_{n=a}^b \left( \frac{nY_n^1}{z^{n+1}} \right) + R'(z) \neq 0, \quad \forall z: R_o \geq |z| \geq R_i,$$

$$X_2 + \sum_{n=a}^b (nX_n^2 z^{n-1}) - \frac{Y_2}{z^2} - \sum_{n=a}^b \left( \frac{nY_n^2}{z^{n+1}} \right) + K_2 R'(z) + I'(z) \neq 0, \quad \forall z: |z| \geq R_o.$$

#### 4. Further discussion

##### 4.1. Complete solutions for a two-phase circular inhomogeneity-matrix system

The solutions presented in this paper, reduce to the case of a two-phase circular inhomogeneity system when the inter-phase layer collapses into the matrix ( $R_i = R_o$ ,  $\alpha_1 = \alpha_2$ ,  $\mu_1 = \mu_2$ ,  $\beta_1 = \beta_2$ ). In this case, from (29) and (30), the solution is given by

$$A_1 = \left[ \alpha_0 \left( 1 - \frac{\mu_0}{\mu_1} \right) + \frac{\mu_0}{\mu_1} \right] C_1 + \left( 1 - \frac{\mu_0}{\mu_1} \right) \frac{\beta_0}{C_1} + iu^*, \quad B_1 = D_1 \left[ 1 - \alpha_1 \left( 1 - \frac{\mu_0}{\mu_1} \right) \right],$$

$$\sum_{n=a}^b A_n z^n = \sum_{n=a}^b \left[ \alpha_0 \left( 1 - \frac{\mu_0}{\mu_1} \right) + \frac{\mu_0}{\mu_1} \right] C_n z^n, \quad \sum_{n=a}^b B_n z^n = \sum_{n=a}^b D_n \left[ 1 - \alpha_1 \left( 1 - \frac{\mu_0}{\mu_1} \right) \right] z^n,$$

since

$$\frac{R_i}{R_o} = 1, \quad K_2 = \left[ \alpha_1 \left( 1 - \frac{\mu_1}{\mu_2} \right) + \frac{\mu_1}{\mu_2} \right] = 1, \quad S_2 = \left( 1 - \frac{\mu_1}{\mu_2} \right) = 0, \quad (\alpha_1 - \alpha_2 K_2) = 0,$$

$$\therefore R_i = R_o, \quad \alpha_1 = \alpha_2, \quad \mu_1 = \mu_2, \quad \beta_1 = \beta_2,$$

which are identical to those established in [15].

##### 4.2. Expressions for $I$ and $J$ inside the inhomogeneity and inside the interphase layer

The principal invariants  $\text{tr}(\mathbf{F}\mathbf{F}^T)$  and  $\det \mathbf{F}$  can be calculated as follows:

$$\text{tr}(\mathbf{F}\mathbf{F}^T) = \text{tr}[F_{ps}F_{qs}(e_p \otimes e_q)] = F_{ps}F_{ps} = y_{p,s}y_{p,s}, \dots \quad p, s = 1, 2,$$

$$\det \mathbf{F} = F_{11}F_{22} - F_{12}F_{21} = y_{1,1}y_{2,2} - y_{1,2}y_{2,1}.$$
(31)

From (31), together with the fact that  $w(z) = y_1(z) + iy_2(z)$ , we have

$$J = -\text{Im}[(w)_{,1} \overline{(w)_{,2}}].$$
(32)

Hence,

$$I = \sqrt{\text{tr}(\mathbf{F}\mathbf{F}^T) + 2J},$$

$$I = \sqrt{(y_{1,1} + y_{2,2})^2 + (y_{1,2} - y_{2,1})^2}.$$
(33)

Eqs. (2) and (24) give us that

$$w_0 = -i\alpha_0 \left[ C_1 z + \sum_{n=a}^b C_n z^n \right] + \bar{D}_1 \bar{z} + \sum_{n=a}^b \bar{D}_n \bar{z}^n - \frac{i\beta_0 z}{\bar{C}_1 + \sum_{n=a}^b n \bar{C}_n \bar{z}^{n-1}}.$$
(34)

Consequently,  $I$  takes the following form within the inhomogeneity

$$I = 2 \left( \alpha_0 \left| C_1 + \sum_{n=a}^b n C_n z^{n-1} \right| + \frac{\beta_0}{|C_1 + \sum_{n=a}^b n C_n z^{n-1}|} \right),$$
(35)

and from Remark 2,

$$|\phi'_0(z)| = \left| C_1 + \sum_{n=a}^b n C_n z^{n-1} \right| \neq 0,$$

$$I^2 - 16\alpha_0\beta_0 = 4 \left( \alpha_0 \left| C_1 + \sum_{n=a}^b n C_n z^{n-1} \right| - \frac{\beta_0}{|C_1 + \sum_{n=a}^b n C_n z^{n-1}|} \right)^2 > 0,$$
(36)



which guarantees that  $F'(I)$ , from (1) is well defined. Similar calculations can be performed for the interphase layer. For example,

$$w_1 = -i\alpha_1 \left[ X_1 z + \sum_{n=a}^b X_n^1 z^n + \frac{Y_1}{z} + \sum_{n=a}^b \frac{Y_n^1}{z^n} + R(z) \right] + \overline{P_1} \bar{z} + \sum_{n=a}^b \overline{P_n^1} \bar{z}^n + \frac{\overline{Q_1}}{\bar{z}} + \sum_{n=a}^b \frac{\overline{Q_n^1}}{\bar{z}^n} + \overline{H(z)} \\ - \frac{i\beta_1 z}{\overline{X_1} + \sum_{n=a}^b (n\overline{X_n^1} z^{n-1}) - \frac{\overline{Y_1}}{z^2} - \sum_{n=a}^b \left( \frac{n\overline{Y_n^1}}{z^{n+1}} \right) + \overline{R'(z)}}, \\ I^2 - 16\alpha_1\beta_1 = 4 \left( \alpha_1 |\phi_1'(z)| - \frac{\beta_1}{|\phi_1'(z)|} \right)^2 > 0,$$

where from Remark 3,

$$|\phi_1'(z)| = \left| X_1 + \sum_{n=a}^b (nX_n^1 z^{n-1}) - \frac{Y_1}{z^2} - \sum_{n=a}^b \left( \frac{nY_n^1}{z^{n+1}} \right) + R'(z) \right| \neq 0,$$

from which  $F'(I)$  is well defined.

#### 4.3. Stress distribution

From [13], the Piola stress distributions for the three-phase circular inhomogeneity-matrix system are given by

$$\begin{aligned} -\sigma_{21} + i\sigma_{11} &= (\chi_0)_{,2}, & \sigma_{22} - i\sigma_{12} &= (\chi_0)_{,1}, & |z| < R_i & \text{(inside the inhomogeneity),} \\ -\sigma_{21} + i\sigma_{11} &= (\chi_1)_{,2}, & \sigma_{22} - i\sigma_{12} &= (\chi_1)_{,1}, & R_o \geq |z| \geq R_i & \text{(inside the interphase layer),} \\ -\sigma_{21}^\infty + i\sigma_{11}^\infty &= (\chi_2^\infty)_{,2}, & \sigma_{22}^\infty - i\sigma_{12}^\infty &= (\chi_2^\infty)_{,1}, & |z| \rightarrow \infty & \text{(at the remote boundary),} \end{aligned} \quad (37)$$

where  $\chi_\gamma(z)$  ( $\gamma = 0, 1, 2$ ), can be determined from (3) as

$$\chi_\gamma(z) = 2\mu_\gamma i \left[ (\alpha_\gamma - 1)\phi_\gamma(z) + i\overline{\psi_\gamma(z)} + \frac{\beta_\gamma z}{\overline{\phi_\gamma'(z)}} \right], \quad \gamma = 0, 1, 2 \text{ (no sum over repeated indices).} \quad (38)$$

Here, functions  $\phi_\gamma(z)$  and  $\psi_\gamma(z)$  are given in (8) and (24)–(26). For example  $\chi_0(z)$  inside the inhomogeneity is given by

$$\chi_0(z) = 2\mu_0 i \left[ (\alpha_0 - 1) \left( C_1 z + \sum_{n=a}^b C_n z^n \right) + i \left( \overline{D_1} \bar{z} + \sum_{n=a}^b \overline{D_n} \bar{z}^n \right) + \frac{\beta_0 z}{(\overline{C_1} + \sum_{n=a}^b n\overline{C_n} z^{n-1})} \right].$$

In addition, the Cauchy stresses [12] inside the inhomogeneity are given by

$$\begin{aligned} \tau_{11} + \tau_{22} &= 2\mu_0 \left[ \frac{IF'(I)}{J} - 2 \right], \\ \tau_{11} - \tau_{22} + 2i\tau_{12} &= 2\mu_0 \frac{F'(I)}{IJ} [(w_0)_{,2}^2 + (w_0)_{,1}^2]. \end{aligned} \quad (39)$$

In our case, from Eqs. (1) and (32)–(36)

$$F'(I) = |\phi_0'(z)| = \left| C_1 + \sum_{n=a}^b nC_n z^{n-1} \right| \neq 0. \quad (40)$$

### 5. Example: The case $a = b = 0$ : uniform remote loading

The general polynomial solutions mentioned above can accommodate many different forms of applied remote loading. Among these, we consider the particular class of uniform remote stress, characterized by the stress functions  $\phi_1(z) = A_1 z$ ,  $\psi_1(z) = B_1 z$ , where  $A_1$  and  $B_1$  are prescribed complex constants.

#### 5.1. Complete solutions

From (8) and (24)–(26), the stress functions at the remote boundary and inside the inhomogeneity and the interphase layer, respectively, are given by

$$\begin{aligned}
\phi_2(z) &= A_1 z, & \psi_2(z) &= B_1 z, & |z| &\rightarrow \infty, \\
\phi_1(z) &= X_1 z + \frac{Y_1}{z} + \frac{S_1 \beta_0 z}{\bar{C}_1}, \\
\psi_1(z) &= P_1 z + \frac{Q_1}{z} + \frac{i \beta_0 R_i^2 (1 - \alpha_1 S_1)}{C_1 z} - \frac{i \beta_1 R_i^2}{z \left( X_1 - \frac{Y_1}{z^2} + \frac{S_1 \beta_0}{\bar{C}_1} \right)}, & R_o \geq |z| \geq R_i, \\
\phi_0(z) &= C_1 z, & \psi_0(z) &= D_1 z, & |z| &< R_i.
\end{aligned} \tag{41}$$

From the general solution (29) and (30), the unknown complex constants  $C_1$  and  $D_1$  are completely determined by the equations

$$\begin{aligned}
A_1 &= \left( K_2 - \frac{\alpha_1 S_2 R_i^2}{R_o^2} \right) \left[ K_1 C_1 + \frac{S_1 \beta_0}{\bar{C}_1} + i u^* \right] + \frac{S_2 R_i^2}{R_o^2} \left[ \alpha_0 C_1 + \frac{\beta_0}{\bar{C}_1} + i u^* \right], \\
B_1 &= \frac{S_1 D_1 R_i^2}{R_o^2} (\alpha_1 - \alpha_2 K_2) + D_1 (1 - \alpha_2 S_2) (1 - \alpha_1 S_1).
\end{aligned} \tag{42}$$

Once  $C_1$  and  $D_1$  are determined,  $X_1$ ,  $Y_1$ ,  $P_1$ , and  $Q_1$  can be calculated from (14) and (16) as

$$\begin{aligned}
X_1 &= K_1 C_1 + i u^*, & Y_1 &= i S_1 \bar{D}_1 R_i^2, \\
P_1 &= D_1 (1 - \alpha_1 S_1), & Q_1 &= i \bar{C}_1 R_i^2 (\alpha_0 - \alpha_1 K_1) + u^* R_i^2 (1 - \alpha_1).
\end{aligned} \tag{43}$$

Also, from Remarks 2–4, we must have that

$$A_1, C_1 \neq 0, \tag{44}$$

$$C_1 \neq 0 \iff \bar{C}_1 \neq 0, \quad \forall z: |z| \leq R_i \text{ (inside the inhomogeneity),} \tag{44}$$

$$X_1 - \frac{Y_1}{z^2} + \frac{S_1 \beta_0}{\bar{C}_1} \neq 0, \quad \forall z: R_o \geq |z| \geq R_i \text{ (inside the interphase layer),} \tag{45}$$

$$X_2 - \frac{Y_2}{z^2} + \frac{K_2 S_1 \beta_0}{\bar{C}_1} + I'(z) \neq 0, \quad \forall z: |z| \geq R_o \text{ (inside the matrix),} \tag{46}$$

where

$$I(z) = \frac{(1 - \alpha_1 S_1) S_2 \beta_0 R_i^2 z}{R_o^2 \bar{C}_1} + \frac{z S_2 \beta_1 (R_o^2 - R_i^2)}{R_o^2 \left( \bar{X}_1 - \frac{\bar{Y}_1 z^2}{R_o^4} + \frac{S_1 \beta_0}{\bar{C}_1} \right)}.$$

**Remark 5 (Eshelby-type result).** We note from (41) that the Piola stress distribution inside the inhomogeneity is uniform, whenever uniform stresses are applied at the remote boundary. We note also from (41) and (43) that the case  $R_i = R_o$ ,  $\alpha_1 = \alpha_2$ ,  $\mu_1 = \mu_2$ ,  $\beta_1 = \beta_2$ , leads to the corresponding solutions from [2] and [14] for the case when the two-phase (circular) inhomogeneity-matrix system is subjected to uniform remote loading.

## 6. Conclusions

Problems relating to three-phase elastic inhomogeneities are of great interest in composite mechanics stemming mainly from the interest in determining the effect of an interphase layer placed between an inhomogeneity (which represents a cross-section of a fibre embedded in a composite material) and its surrounding matrix. In this paper we consider exactly this scenario but for a class of materials undergoing *finite* plane deformations. Specifically, we consider a three-phase circular inhomogeneity-matrix system from a particular class of compressible hyperelastic materials of harmonic-type undergoing *finite* plane deformations. The intermediate layer between the inhomogeneity and the matrix is assumed to have its own separate material properties allowing for the possibility of incorporating material bonding effects not accommodated by the traditional two-phase perfect interphase model. We obtain the complete solution when the three-phase system is subjected to classes of nonuniform remote stresses characterized by stress functions described by general polynomials of degree  $n$ . As a particular case of the aforementioned analysis, we obtain an Eshelby-type result that a three-phase circular inhomogeneity under uniform remote stress and eigenstrain, admits an internal uniform stress field when subjected to plane (finite) deformations. We mention also that, when used in conjunction with the method of conformal mapping, the above techniques can be extended to accommodate the analysis of systems which include a noncircular three-phase inhomogeneity-matrix system whenever the corresponding conformal mapping function is known (for example, an elliptic inhomogeneity).

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